Int J Solids Structures Vol 18, No 3, pp 249-261, 1982
Printed in Great Britain

0020-7683/82/030249-13$03 00/0
Pergamon Press Ltd

DYNAMIC RESPONSE OF AN EMBEDDED RECTANGULAR FOUNDATION TO ANTIPLANE SHEAR WAVES†

KEVIN P. MEADE and LEON M. KEER
Northwestern University, Evanston, IL 60201, U.S A.

(*Received* 3 *February* 1981; *in revised form* 31 *July* 1981)

Abstract—The steady state response of a deep, narrow, rectangular foundation to antiplane shear waves is studied. Consideration is given first to the dynamic response of a single rigid line inclusion This result is used to model a rectangular foundation in a manner similar to Haritos and Keer who studied the problem of a rigid block partially embedded in an elastic half-space. Because the problem is antiplane, the results arrived at for the deep, narrow, rectangular foundation also apply to a shallow, wide, rectangular foundation.

INTRODUCTION

In recent years the problem of designing earthquake resistant structures has become increasingly important. Fundamental to this problem is understanding the interaction between the soil and structure. Efforts have proceeded in two general directions. One method adapts the finite element method to solve wave propagation problems for semi-infinite domains [1]. The results can be used to estimate dynamic compliances for fairly complex foundation geometries and a wide range of soil properties [2]. Another line of attack makes use of analytical tools such as the boundary integral method, the integral transform method, and the Green's function method. More idealized foundation shapes are treated (e.g. semi-cylindrical [3, 4], rectangular [5–7], semi-elliptical [8], strip [9–12]) with these methods. However, although limited, they offer some advantages over the finite element method in that results can often lead to exact or asymptotic solutions and offer no difficulty in satisfying the radiation conditions inherent in wave propagation problems for semi-infinite domains [1]. The development of analytical expressions for soil-structure interaction is important because, as demonstrated by Lee and Trifunac [13], only analytical results can provide a complete check on approximate methods used in the solution of a problem.

In this paper the soil-structure interaction problem for a deep, narrow rectangular foundation is studied (Fig. 1). The soil is represented by a homogeneous, isotropic, linearly elastic half-space and the loading on the foundation is represented by antiplane shear waves with harmonic time dependence. The foundation is represented by rigid inclusions in a manner similar to Haritos and Keer [14].

The paper is organized as follows. First, the equations of motion for a rigid line inclusion are formulated using the procedure of Thau [15]. Next, making use of this result, the equations of motion for a deep, narrow rectangular foundation are given and the associated assumptions and approximations are explained. The numerical procedure used to solve the equations is described and finally the numerical results are discussed.

Because of the nature of the antiplane formulation of this problem, similar results and conclusions arrived at for a deep, narrow rectangular foundation also apply to a shallow, wide rectangular foundation.

RIGID LINE INCLUSION

Formulation

The geometry and coordinate system are depicted in Fig. 2. A rigid line inclusion is embedded a unit depth in an elastic half-space with density ρ and shear modulus μ. The surface of the half-space is free of tractions. A state of antiplane strain is treated in which $u_z = u_z(x, y, t)$ is the only non-zero displacement component. This represents the limiting case for the deep, narrow rectangular foundation, i.e. $h/a = 0$.

†The authors are grateful for support from the National Science Foundation, grant CME 7918015.

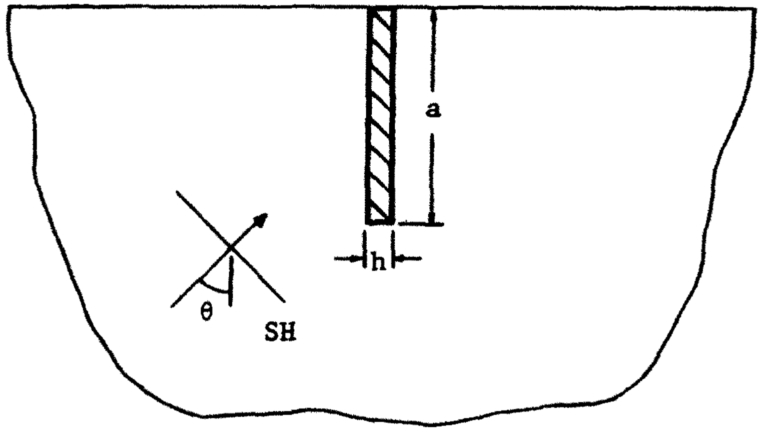

Fig. 1. Foundation geometry.

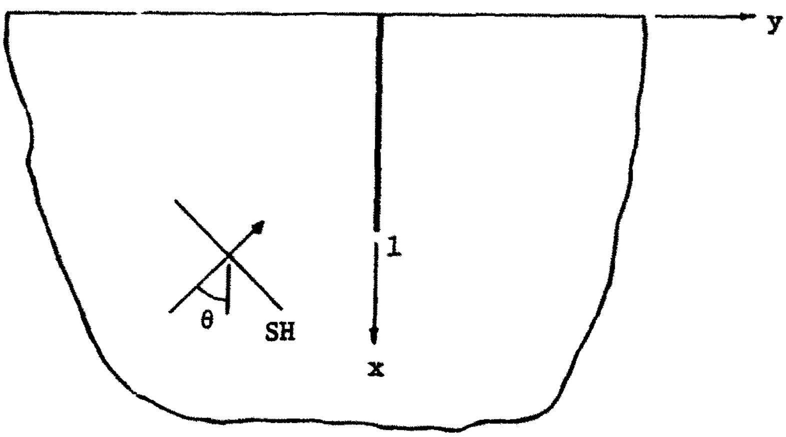

Fig. 2. Rigid line inclusion.

Incident upon the inclusion is a plane time harmonic shear wave with amplitude unity. It has the form

$$u_z^{(i)}(x, y, t) = w^{(i)}(x, y)\,e^{-i\omega t} = e^{ik(-x\cos\theta + y\sin\theta)}\,e^{-i\omega t} \tag{1}$$

where k is the dimensionless wave number, ω is the frequency, and θ is the angle of incidence.

Since the time harmonic response of the inclusion, $\Delta\,e^{-i\omega t}$ say, is to be calculated, the total field is written as

$$u_z(x, y, t) = w^{(i)}(x, y)\,e^{-i\omega t} + w^{(s)}(x, y)\,e^{-i\omega t} \tag{2}$$

where $w^{(s)}(x, y)\,e^{-i\omega t}$ is the scattered field satisfying Helmholtz' equation,

$$\nabla^2 w^{(s)} + k^2 w^{(s)} = 0 \tag{3}$$

boundary conditions,

$$[w^{(i)}(x, 0) + w^{(s)}(x, 0)]\,e^{-i\omega t} = \Delta\,e^{-i\omega t} \qquad (0 < x < 1) \tag{4}$$

and

$$\left[\mu\frac{\partial w^{(i)}}{\partial x}(0, y) + \mu\frac{\partial w^{(s)}}{\partial x}(0, y)\right]e^{-i\omega t} = 0 \qquad |y| < \infty \tag{5}$$

as well as the usual radiation conditions. In the sequel, the factor $e^{-i\omega t}$ is dropped.

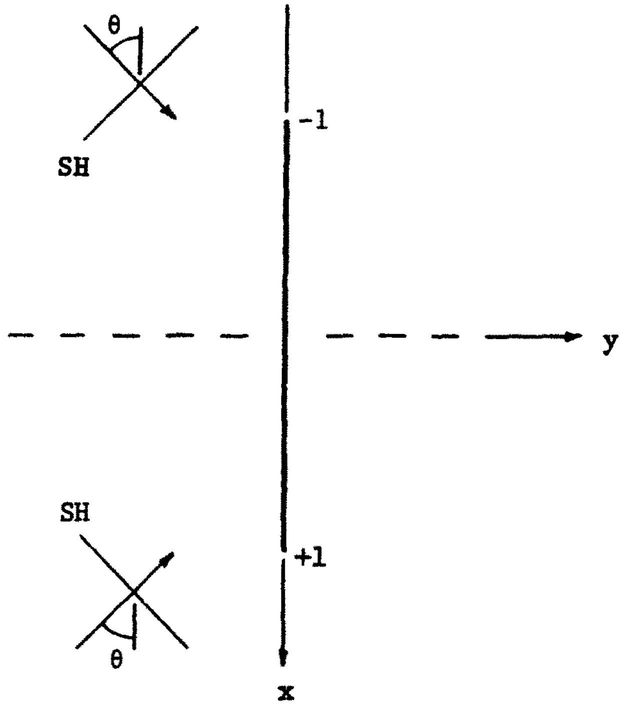

Fig. 3. Rigid line inclusion: equivalent full-space problem.

The equivalent full-space problem is depicted in Fig. 3. Here the "incident" field takes the form

$$w^{(i+r)}(x, y) = e^{iky \sin\theta} 2 \cos(kx \cos\theta). \tag{6}$$

Note that the above automatically satisfies the zero traction condition on the surface of the half-space. Equation (6) represents the incident field plus the wave reflected from the half-space boundary in the absence of the line inclusion.

By extending the domain of $w^{(s)}(x, y)$ to $x < 0$, the field for the full-space problem is

$$w(x, y) = w^{(i+r)}(x, y) + w^{(s)}(x, y) \tag{7}$$

where

$$w^{(s)}(x, y) = w^{(s)}(-x, y) \tag{8}$$

and

$$\nabla^2 w + k^2 w = 0 \tag{9}$$

with

$$w(x, 0) = \Delta \qquad |x| < 1. \tag{10}$$

Equation (8) guarantees that the zero traction condition is identically satisfied.

Method of solution

Following Thau[15] the scattered field is calculated in two parts: the diffracted field $w_1(x, y)$ and the radiated field $w_2(x, y)$. Thus

$$w(x, y) = w^{(i+r)}(x, y) + w_1(x, y) + \Delta w_2(x, y) \tag{11}$$

where $w_1(x, y)$ and $w_2(x, y)$ are determined by solving

$$\nabla^2 w_j + k^2 w_j = 0, \qquad j = 1, 2 \tag{12}$$

with

$$w^{(i+r)}(x, 0) + w_1(x, 0) = 0 \qquad |x| < 1 \tag{13}$$

$$w_2(x, 0) = 1 \qquad |x| < 1. \tag{14}$$

It is important to emphasize that $w_1(x, y)$ is the field representing the diffraction of the incident field by the inclusion held fixed while $w_2(x, y)$ represents the radiated field from the inclusion translating with displacement $\Delta\, e^{-i\omega t}$ in the absence of an incident field. The resultant z-directed force from these two problems will be equated to $-\mu k^2 M_F \Delta$ where M_F is the normalized mass per unit length associated with the inclusion.

The following representation is used for both fields:

$$w_j(x, y) = -\frac{i}{4}\int_{-1}^{1} b_j(s) H_0^{(1)}(k\sqrt{((s-x)^2 + y^2)})\, \mathrm{d}s, \qquad (j = 1, 2) \tag{15}$$

where

$$b_j(s) = \frac{\partial w_j}{\partial y}(s, 0^+) - \frac{\partial w_j}{\partial y}(s, 0^-) \tag{16}$$

is proportional to the jump in shear stress across the inclusion[17] and $H_0^{(1)}(\cdot)$ is a Hankel function of the first kind of order zero[18]. Equation (15) was derived using integral transform techniques[16] although other methods will yield an identical result[17].

The resultant z-directed forces from the two problems are given by

$$F_j = \mu \int_{-1}^{1} b_j(s)\, \mathrm{d}s \qquad (j = 1, 2) \tag{17}$$

where F_1 is the driving force and F_2 the impedance.

Equation (15) is integrated by parts to yield the result

$$w_j(x, y) = -\frac{i}{8} B_j(1)[H_0^{(1)}(k\sqrt{((1+x)^2 + y^2)}) + H_0^{(1)}(k\sqrt{((1-x)^2 + y^2)})]$$
$$-\frac{i}{8}\int_{-1}^{1} B(s) \frac{k(s-x)}{\sqrt{((s-x)^2 + y^2)}} H_1^{(1)}(k\sqrt{((s-x)^2 + y^2)})\, \mathrm{d}s \qquad (j = 1, 2) \tag{18}$$

where

$$B_j(s) = \int_{-s}^{s} b_j(\xi)\, \mathrm{d}\xi \qquad (j = 1, 2) \tag{19}$$

$$B_1(1) = \mu^{-1} F_1 \qquad B_2(1) = \Delta^{-1}\mu^{-1} F_2 \tag{20}$$

and $H_1^{(1)}(\cdot)$ is the Hankel function of the first kind of order one[18].

Substituting eqn (19) into eqns (13) and (14) gives

$$\frac{iB_j(1)}{2}[H_0^{(1)}(k|1+x|) + H_0^{(1)}(k|1-x|)]$$
$$+\frac{1}{\pi}\int_{-1}^{1} B_j(s)\left[\frac{1}{s-x} + \frac{i\pi k}{2}\operatorname{sgn}(s-x) N(k|s-x|)\right] \mathrm{d}s = -4 f_j(x) \tag{21}$$

where

$$f_1(x) = -w^{(i+r)}(x,0), \tag{22}$$

$$f_2(x) = 1, \tag{23}$$

and

$$N(k|s-x|) = H_1^{(1)}(k|s-x|) + 2i/\pi k|s-x|. \tag{24}$$

Upon solution of these Cauchy integral equations, Δ is determined from

$$-\mu k^2 M_F \Delta = F_1 + \Delta F_2. \tag{25}$$

Therefore

$$|\Delta| = \left|\frac{F_1}{\mu k^2 M_F + F_2}\right|. \tag{26}$$

Numerical solution

In order to obtain a numerical solution, eqn (21) is modified by the transformation

$$B_j(s) = sB_j(1) + \phi_j(s) \qquad (j = 1,2) \tag{27}$$

where $\phi_j(s)$ can be shown to exhibit the properties

$$\phi_j(s) = -\phi_j(-s) \tag{28}$$

$$\phi_j(s) = (1-s^2)^{1/2}\psi_j(s) \tag{29}$$

$$|\psi_j(s)| < \infty \qquad |s| \le 1. \tag{30}$$

The result is

$$\frac{i}{2}B_j(1)\int_{-1}^{1} H_0^{(1)}(k|s-x|)\,\mathrm{d}s + \frac{1}{\pi}\int_{-1}^{1}\phi_j(s)\left[\frac{1}{s-x} + \frac{ik\pi}{2}\operatorname{sgn}(s-x)N(k|s-x|)\right]\mathrm{d}s = -4f_j(x) \tag{31}$$

$$(j = 1,2) \qquad |x| < 1.$$

The above equation is solved using the collocation scheme of Erdogan and Gupta [19]. Upon application of the Gauss-Chebyshev integration formula we obtain the system of algebraic equations

$$\sum_{q=1}^{N}\frac{1-t_q^2}{N+1}\psi_j(t_q)\left[\frac{1}{x_p-t_q} + \pi\kappa(t_q,x_p)\right] + \frac{iB_j(1)}{2}\int_{-1}^{1} H_0^{(1)}(k|s-x_p|)\,\mathrm{d}s = -4f_j(x_p) \tag{32}$$

where

$$p = 1, 2, \ldots, N+1$$

$$x_p = \cos\left[\pi(2p-1)/2(N+1)\right] \tag{33}$$

$$t_q = \cos\left[\pi q/(N+1)\right] \tag{34}$$

$$\kappa(t_q, x_p) = \frac{i\pi k}{2}\operatorname{sgn}(t_q - x_p)N(k|t_q - x_p|). \tag{35}$$

The system provides $N+1$ equation for the $N+1$ unknowns $\psi_j(t_1), \ldots, \psi_j(t_N)$, and $B_j(1)$. In the present computations, N was chosen as an even integer. The integral of the Hankel function was evaluated using a seven point Lagrange interpolation scheme of a table in Ref. [18].

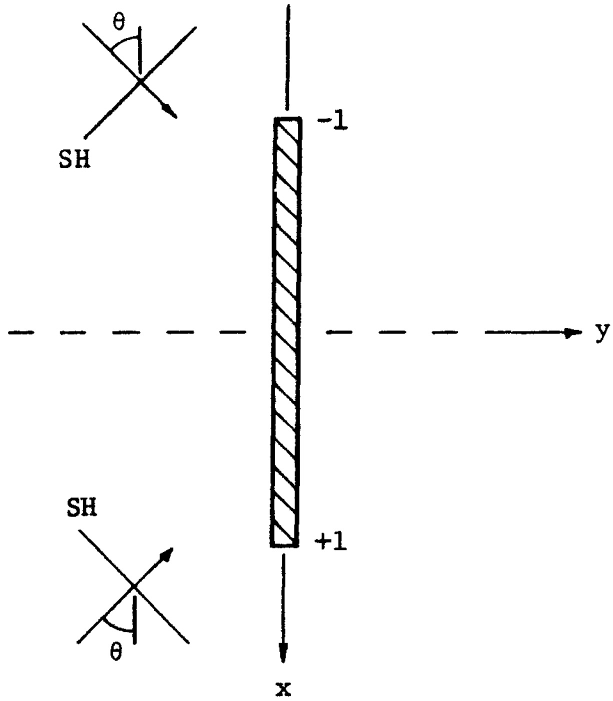

Fig. 4. Foundation of finite thickness: equivalent full-space problem.

FOUNDATION OF FINITE THICKNESS

The results of the previous section are now used to formulate the problem of a deep, narrow, rectangular foundation embedded a unit depth in an elastic half-space. The thickness to depth ratio is equal to ϵ where $0 < \epsilon \ll 1$ and the incident field is the same as before.

The boundary conditions for the equivalent full-space problem (Fig. 4) are

$$w\left(x, -\frac{\epsilon}{2}\right) = \Delta \qquad |x| < 1 \tag{36}$$

$$w\left(x, \frac{\epsilon}{2}\right) = \Delta \qquad |x| < 1 \tag{37}$$

$$w(1, y) = \Delta \qquad |y| < \epsilon/2 \tag{38}$$

$$w(-1, y) = \Delta \qquad |y| < \epsilon/2. \tag{39}$$

The foundation is represented by two rigid line inclusions placed a distance ϵ apart. Along these inclusions boundary conditions (36) and (37) are imposed. Boundary conditions (38) and (39) are not satisfied. However, this is not a great disadvantage since we wish to calculate impedance functions for the foundation which are "global" properties. The boundary conditions are violated "locally," i.e. in the neighborhood of $x = \pm 1$, to simplify the analytical treatment of the problem. This is similar to the procedure used by Haritos and Keer[14] to analyze the problem of an elastic half-space in which a perfectly bonded, rigid rectangular block is partially embedded.

An additional consideration is that the material between the two inclusions should move as a rigid body. Asymptotic analysis of the interior problem for the Helmholtz equation in a long, narrow, rectangular strip with boundary conditions (36) and (37) and arbitrary boundary conditions at the ends reveals that for low frequencies $w \sim \Delta$ in the strip and deviates from that only in the boundary layer of thickness $O(\epsilon/2)$ near the ends of the strip (see Appendix). Thus, for low frequencies ($(k\epsilon)^2 \ll 1$) this model will give adequate results for the determination of the impedance functions and the time harmonic response of the foundation.

The total field is written as

$$w(x, y) = w^{(i+r)}(x, y) + w_1^-(x, y) + w_1^+(x, y) + \Delta w_2^-(x, y) + \Delta w_2^+(x, y) \tag{40}$$

where

$$w_j^{\mp}(x,y) = -\frac{i}{4}\int_{-1}^{1} b_j^{\mp}(s)H_0^{(1)}\left(k\sqrt{\left((s-x)^2+\left(y\pm\frac{\epsilon}{2}\right)^2\right)}\right)\mathrm{d}s, \quad (j=1,2). \tag{41}$$

The unknowns are determined from

$$w^{(i+r)}\left(x,-\frac{\epsilon}{2}\right)+w_1^-\left(x,-\frac{\epsilon}{2}\right)+w_1^+\left(x,-\frac{\epsilon}{2}\right)=0 \quad |x|<1 \tag{42}$$

$$w^{(i+r)}\left(x,\frac{\epsilon}{2}\right)+w_1^-\left(x,\frac{\epsilon}{2}\right)+w_1^+\left(x,\frac{\epsilon}{2}\right)=0 \quad |x|<1 \tag{43}$$

$$w_2^-\left(x,-\frac{\epsilon}{2}\right)+w_2^+\left(x,-\frac{\epsilon}{2}\right)=1 \quad |x|<1 \tag{44}$$

$$w_2^-\left(x,\frac{\epsilon}{2}\right)+w_2^+\left(x,\frac{\epsilon}{2}\right)=1 \quad |x|<1. \tag{45}$$

As before, eqn (41) is integrated by parts and substituted into eqns (42)–(45). The resulting system of Cauchy integral equations can be uncoupled. The uncoupled equation takes the form:

$$\begin{aligned} &i\beta_j^{\pm}(1)\left\{\int_{-1}^{1} H_0^{(1)}(k|s-x|)\,\mathrm{d}s \pm \int_{-1}^{1} H_0^{(1)}(k\sqrt{((s-x)^2+\epsilon^2)})\,\mathrm{d}s\right\} \\ &\quad+\frac{1}{\pi}\int_{-1}^{1}\phi_j^{\pm}(s)\left[\frac{1}{s-x}+\frac{i\pi k}{2}\operatorname{sgn}(s-x)N(k|s-x|)\right. \\ &\quad\left.\pm\frac{i\pi k(s-x)}{2\sqrt{((s-x)^2+\epsilon^2)}}H_1^{(1)}(k\sqrt{((s-x)^2+\epsilon^2)})\right]\mathrm{d}s=-4f_j^{\pm}(x) \end{aligned} \tag{46}$$

where

$$\beta_j^{\pm}(1)=\frac{1}{2}(B_j^{\mp}(1)\pm B_j^{\mp}(1)) \qquad (j=1,2)$$

$$f_1^{\pm}(x)=w^{(i+r)}\left(x,-\frac{\epsilon}{2}\right)\pm w^{(i+r)}\left(x,\frac{\epsilon}{2}\right)$$

$$f_2^{+}(x)=2, \qquad f_2^{-}(x)=0.$$

The definition of the $B_j^{\mp}(1)$'s is the same as before. Also, the numerical scheme described in the previous section is applicable here without modification.

Upon solution of the integral equations, Δ is determined from

$$-\mu k^2 M_F\Delta = F_1^- + F_1^+ + \Delta(F_2^- + F_2^+) \tag{47}$$

where

$$M_F = M_S + M_E \tag{48}$$

$$F_1^- + F_1^+ = \mu[B_1^-(1) + B_1^+(1)] \tag{49}$$

$$F_2^- + F_2^+ = -\mu k^2 M_S + \mu[B_2^-(1) + B_2^+(1)]. \tag{50}$$

The quantity M_S is the ratio of the mass per unit length of the foundation made of soil to the mass per unit length of a unit cube of soil. The quantity M_E is the ratio of the mass per unit length of the foundation to the mass per unit length of a unit cube of soil that is in excess of M_S. Thus, M_F represents the ratio of the total mass per unit length of the foundation to the mass per unit length of a unit cube of soil.

Equation (49) gives the driving force and eqn (50) gives the impedance for the massless foundation.

Therefore

$$|\Delta| = \left| \frac{F_1^- + F_1^+}{\mu k^2 M_F + F_2^- + F_2^+} \right|. \tag{51}$$

RESULTS

Figures 5 and 6 give the dynamic response of a single rigid line inclusion. The amplitude of motion $|\Delta|$ is plotted versus the dimensionless wavenumber k. The quantity M_F is the mass per

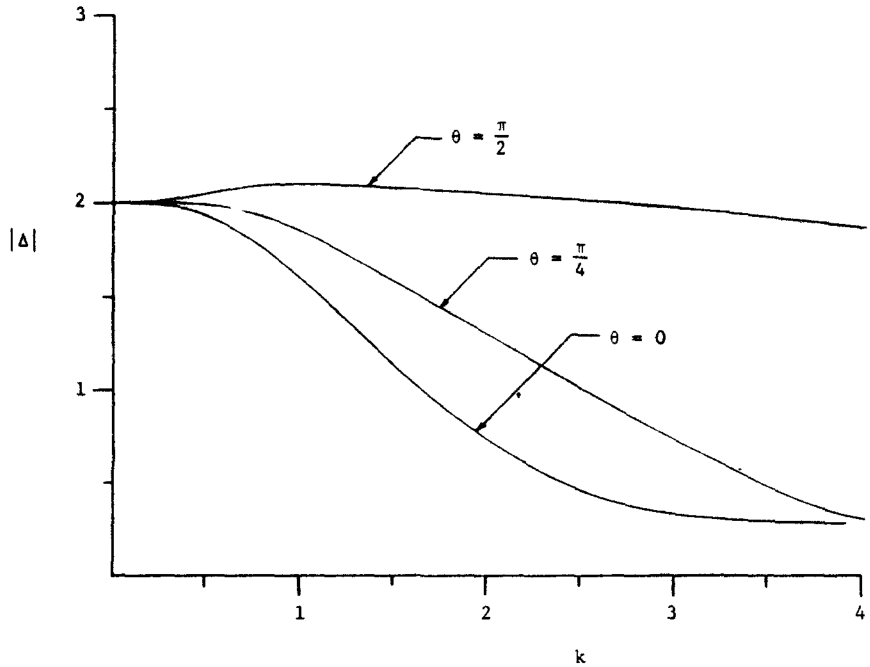

Fig. 5. Response of rigid line inclusion $M_F = 0.25$

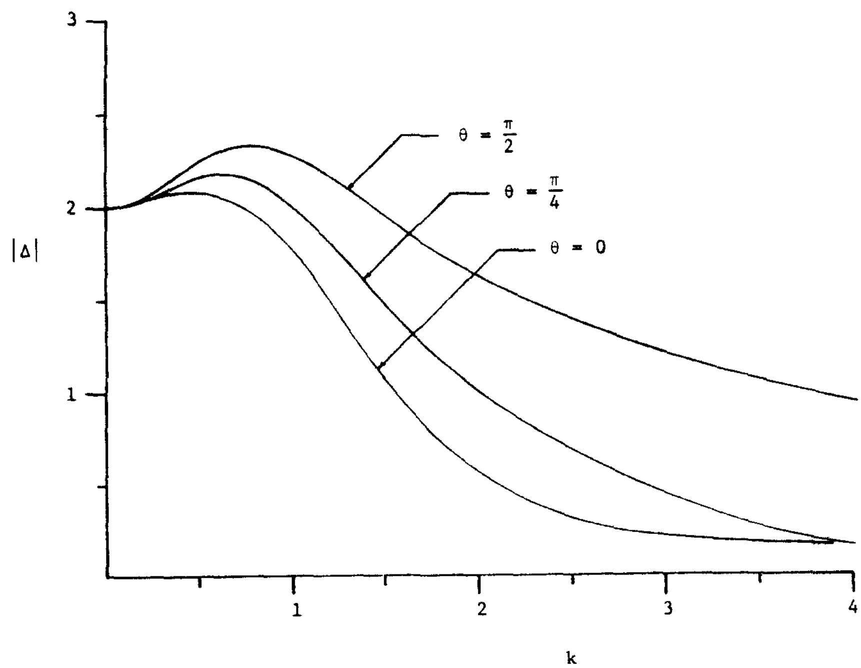

Fig 6. Response of rigid line inclusion $M_F = 1\ 00$

unit length of the inclusion normalized with respect to the mass per unit length of a unit cube of soil. The plots are for $M_F = 0.25$ and 1.00 and $\theta = 0$, $\pi/4$, and $\pi/2$.

Figure 5 shows that for $M_F = 0.25$ and $\theta = \pi/2$ (grazing incidence) the inclusion moves essentially like the half-space would move in the absence of the inclusion. For the other angles of incidence, $|\Delta|$ is seen to change significantly with changes in k.

Figure 6 shows that $|\Delta|$ departs significantly from the free-field amplitude (equal to 2) for certain values of k. This can be understood in terms of an equivalent single degree of freedom system. Since the embedment depth is fixed, the equivalent elastic spring and dashpot constants are fixed. Therefore increasing the mass results in the reduction of the equivalent natural frequency and fraction of critical damping[8].

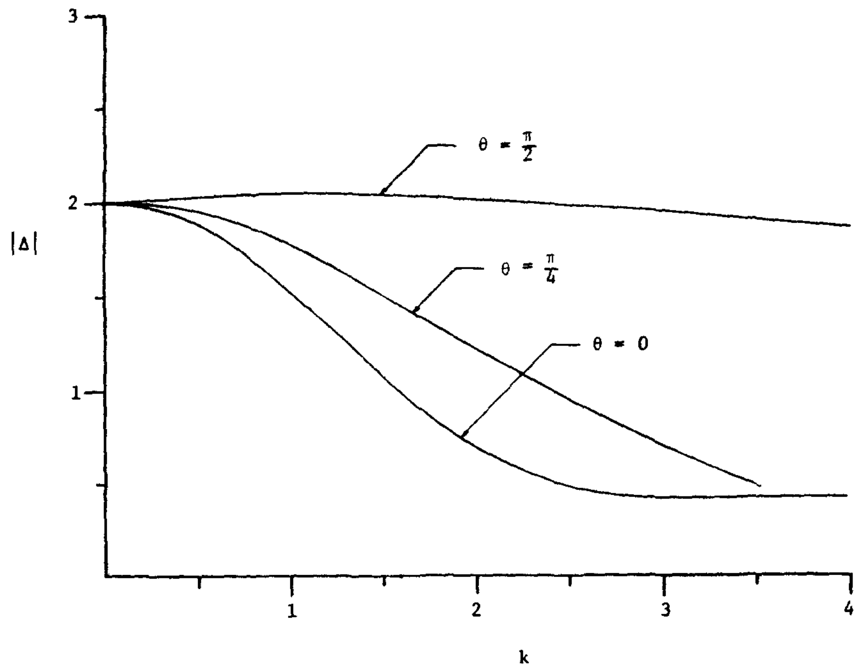

Fig. 7. Foundation response $M_F = 0.25$, $\epsilon = 0.1$.

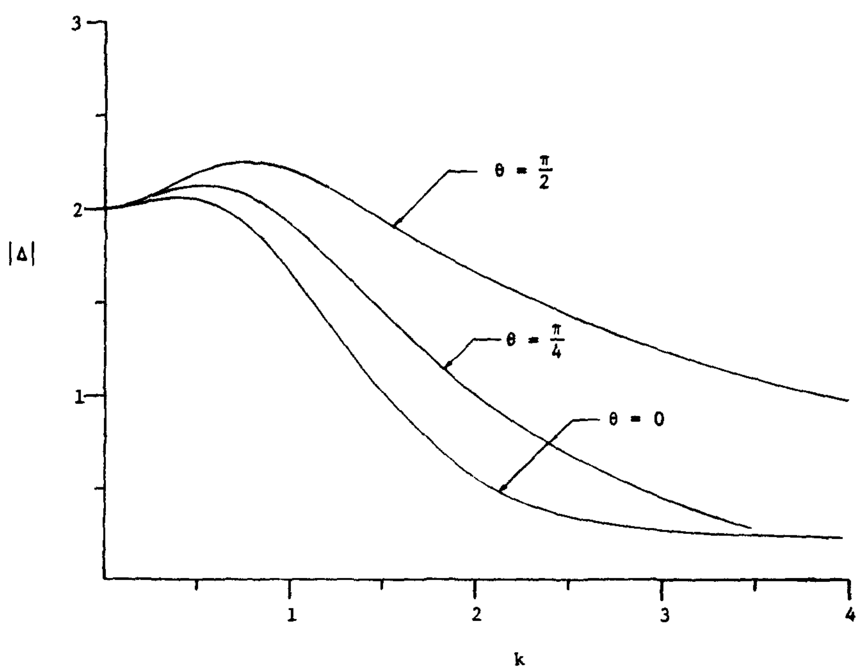

Fig. 8. Foundation response $M_F = 1.00$, $\epsilon = 0.1$

Figures 7 and 8 give the dynamic response of a foundation of finite thickness. As before, the amplitude of motion $|\Delta|$ is plotted versus the dimensionless wavenumber k. The cases reported here are for $M_F = 0.25$ and 1.00, $\epsilon = 0.1$ and $\theta = 0$, $\pi/4$, and $\pi/2$.

The results for the foundation of finite thickness exhibit characteristics similar to those for the rigid line inclusion. Figure 7 shows that for $M_F = 0.25$, $\epsilon = 0.1$, and $\theta = \pi/2$ (grazing incidence) the foundation moves essentially like the half-space in the absence of the foundation.

Figure 8 shows that $|\Delta|$ can depart significantly from the free-field motion. This can also be understood in terms of an equivalent single degree of freedom system. The results obtained here are similar to those of Wong and Trifunac[8] for a semi-elliptical foundation whose minor axis to major axis ratio is 0.05.

Figures 9 and 10 show the stiffness and damping functions, respectively. The stiffness for

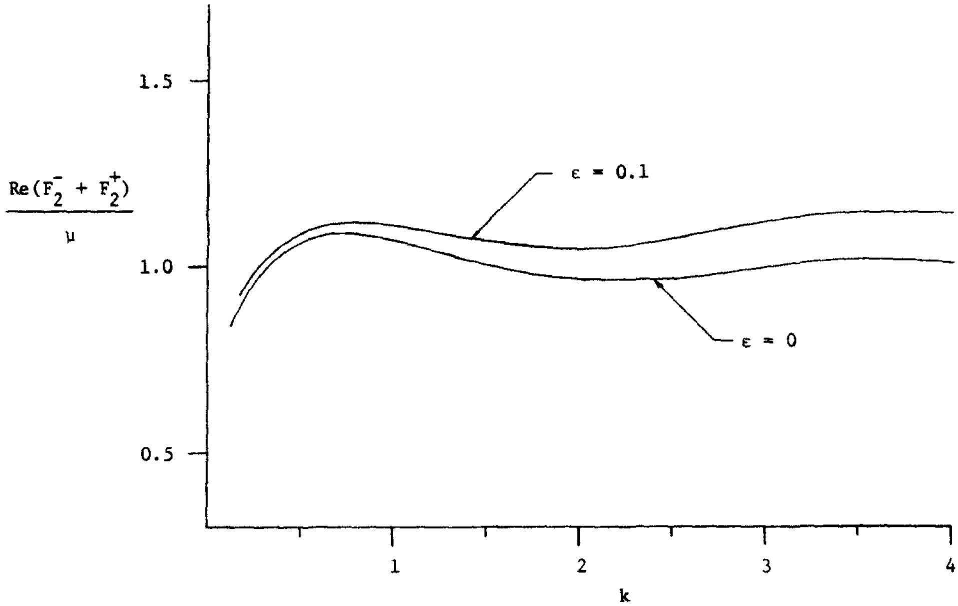

Fig. 9 Stiffness function.

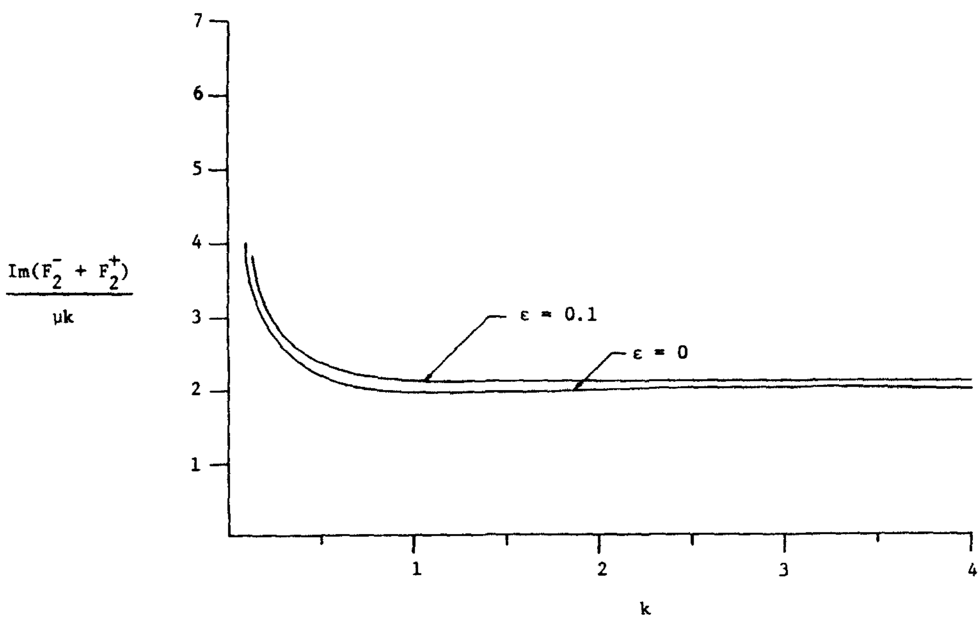

Fig 10. Damping function.

the foundation of finite thickness is greater than that of the single rigid line inclusion and approaches a value of 1.15 for $k = 4$. This is consistent with the findings of a number of other investigators[3–8, 12]. In particular, the results of Hradilek[12] for the stiffness and damping functions agree with those shown in Figs. 9 and 10 for $\epsilon = 0$. The damping is also greater for the finite thickness foundation. This explains the fact that the peak response of the single inclusion is greater than that of the foundation of finite thickness.

CONCLUSION

In this paper the dynamic response of a deep, narrow, rectangular foundation to antiplane shear waves has been determined. These results also apply to the case of a shallow, wide, rectangular foundation because of the nature of the antiplane formulation. The results were found to be similar to those obtained by Wong and Trifunac[8] and Hradilek[12].

Similar considerations used in the solution of this antiplane problem can be applied to the corresponding in-plane problem. Also, the results obtained here can be utilized in analyzing the dynamic response of extended structures, i.e. structures that are supported on more than one foundation.

REFERENCES

1 G. Dasgupta, Foundation impedance matrices by substructure deletion. *J. EMD, Proc. ASCE* **106**, EM3, 517 (1980).
2 M. C Chen and J Penzine, An investigation of the effectiveness of existing bridge design methodology in providing adequate structural resistance to seismic disturbances. Phase III. Nonlinear soil-structure interaction of skew highway bridges. Federal Highway Adminstration, *Rep No FHWA-RD*-77-169 (1977).
3 J E Luco, Dynamic interaction of a shear wall with the soil *J. Engng Mech. Div. ASCE* **95**, 333–346 (1969).
4 M. D Trifunac, Interaction of a shear wall with the soil for incident plane SH-waves. *Bull. Seismological Soc Am.* **62**, 63–83 (1972)
5 S A Thau and A. Umek, Transient response of a buried foundation to antiplane shear waves. *J. Appl Mech. ASME* **95**, 1061–1066 (1973).
6 S. A Thau and A Umek, Coupled rocking and translating vibrations of a buried foundation. *J. Appl. Mech. ASME* **41**, 697–702 (1974)
7 H. L. Wong, Dynamic soil structure interaction. *Rep EERL*-75-01, Earthq. Eng. Res. Lab., California Institute of Technology, Pasadena, California (1975)
8. H L. Wong and M. D. Trifunac, Interaction of a shear wall with the soil for incident plane SH-waves elliptical rigid foundation *Bull Seismologic Soc Am.* **64**, 1825–1842 (1974)
9. D. D. Ang and L. Knopoff, Diffraction of scalar waves by a clamped finite strip. *Proc. Nat. Acad. Sci.* **51**, 471 (1964).
10. P. Karasudhi, L M. Keer and S. L. Lee, Vibratory motion of a body on an elastic half-plane. *J. Appl. Mech. ASME*, 1–9 (1968)
11. J. E. Luco and R. A. Westmann, Dynamic response of a rigid footing bonded to an elastic half-space. *J. Appl. Mech., ASME* **39**, 527–534 (1972)
12 P. J. Hradilek, Dynamic soil structure interaction M. S thesis, School of Engineering and Applied Science, UCLA (1970).
13 V. W Lee and M. D. Trifunac, Response of tunnels to SH-waves, *J. EMD, Proc ASCE* **105**, EM4, 643 (1979).
14 G. K. Haritos and L. M Keer, Stress analysis for an elastic half-space containing an embedded rigid block *Int. J. Solids Structures* **16**, 19–40 (1980).
15 S. A. Thau, Radiation and scattering from a rigid inclusion in an elastic medium. *J. Appl. Mech.* 509–511 (1967).
16. I. N. Sneddon, *The Use of Integral Transforms.* McGraw-Hill New York (1972)
17 Y. H. Pao and C C Mow, *Diffraction of Elastic Waves and Dynamic Stress Concentrations*, 1st Edn. Crane-Russak, New York (1973).
18. M. Abramowitz and I A. Stegun (Eds), *Handbook of Mathematical Functions.* Dover, New York (1968).
19. F. Erdogan and G D Gupta, On the numerical solution of singular integral equations. *Quart. Appl. Math.* **30**, 525 (1972)
20. A H Nayfeh, *Perturbation Methods.* Wiley, New York (1953).

APPENDIX

Consider the following boundary value problem for Helmholtz' equation in a thin domain:

$$\bar{u}_{XX} + \bar{u}_{YY} + k^2\bar{u} = 0 \tag{A1}$$

$$\bar{u}(X, 0) = 1 \qquad 0 < X < L \tag{A2}$$

$$\bar{u}(X, h) = 1 \qquad 0 < X < L \tag{A3}$$

$$\bar{u}(0, Y) = G_1(Y) \qquad 0 < Y < h \tag{A4}$$

$$\bar{u}(L, Y) = G_2(Y) \qquad 0 < Y < h. \tag{A5}$$

The geometry is depicted in Fig. 11 This domain is representative of the interior of the foundation in the full-space problem where $\bar{u} = w/\Delta$ and $L = 2a$.

Under the transformation $x = X/L$, $y = Y/h$, $G_1(Y) = g_1(y)$, $G_2(Y) = g_2(y)$, and $\bar{u}(X, Y) = u(x, y)$ the boundary value

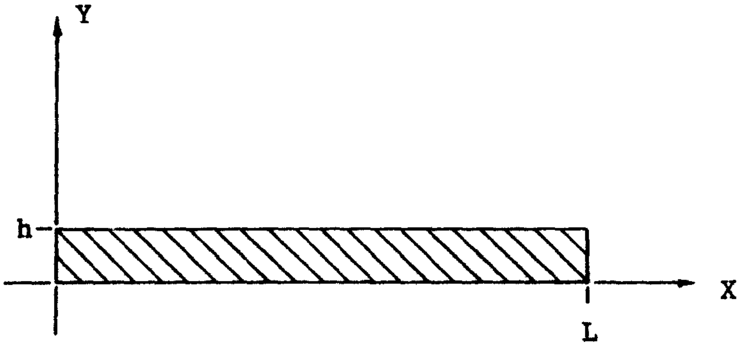

Fig. 11 Geometry of thin domain.

problem becomes:

$$\eta^2 u_{xx} + u_{yy} + \eta^2\beta^2 u = 0 \tag{A6}$$

$$u(x, 0) = 1 \qquad 0 < x < 1 \tag{A7}$$

$$u(x, 1) = 1 \qquad 0 < x < 1 \tag{A8}$$

$$u(0, y) = g_1(y) \qquad 0 < y < 1 \tag{A9}$$

$$u(1, y) = g_2(y) \qquad 0 < y < 1 \tag{A10}$$

where

$$\eta^2 = h^2/L^2 \ll 1 \tag{A11}$$

and we require

$$\beta^2 = k^2L^2 = 0(1) \tag{A12}$$

In order to construct an asymptotic expansion of the solution we take an ansatz of the form:

$$u \sim u^0 + \eta^2 u^2 + \eta^4 u^4 + \cdots \tag{A13}$$

Substituting this into eqns (A6) through (A10) gives

$$u^0_{yy} = 0 \qquad u^0(x, 0) = 1 \qquad u^0(x, 1) = 1 \tag{A14}$$

$$u^n_{yy} = -u^{n-2}_{xx} - \beta^2 u^{n-2} \qquad u^n(x, 0) = 0 \qquad u^n(x, 1) = 0 \tag{A15}$$

where n is an even number greater than or equal to two.

Solution of the above shows

$$u \sim 1 + 0(\eta^2) \tag{A16}$$

We note that this solution will not, in general, satisfy eqns (A9) and (A10). Therefore, this so-called "outer expansion" must be modified by the "boundary layer expansion" near the ends of the domain[20].

Because of the nature of the antiplane formulation of the title problem we lose no generality by considering $g_1(y) = g_2(y) = g(y)$. In this case the boundary layer construction is identical for both ends.

To construct the boundary layer expansion near $x = 0$, we introduce the stretching transformation

$$\xi = x/\eta. \tag{A17}$$

The boundary layer solution, $U(\xi, y)$, satisfies:

$$U_{\xi\xi} + U_{yy} + \eta^2\beta^2 U = 0 \tag{A18}$$

$$U(0, y) = g(y) \qquad 0 < y < 1 \tag{A19}$$

$$U(\xi, 0) = 1 \qquad 0 < \xi < \infty \tag{A20}$$

$$U(\xi, 1) = 1 \qquad 0 < \xi < \infty \tag{A21}$$

In addition to satisfying eqns (A18)–(A21) the boundary layer solution must be "matched" with the outer expansion. Since, for simplicity, we are considering here only leading order terms it is sufficient to utilize the matching procedure of Prandtl[20], i.e. we require

$$\lim_{\xi\to\infty} U(\xi, y) = \lim_{x\to 0} u(x, y) \tag{A22}$$

where $u(x, y)$ is given by eqn (A16)

The ansatz for the inner expansion is

$$U \sim U^0 + \eta^2 U^2 + \eta^4 U^4 + \quad \text{(A23)}$$

which yields

$$U^0_{\xi\xi} + U^0_{yy} = 0 \quad U^0(0, y) = g(y) \quad U^0(\xi, 0) = U^0(\xi, 1) = 1 \quad \text{(A24)}$$

$$\lim_{\xi\to\infty} U^0(\xi, y) = 1$$

$$U^n_{\xi\xi} + U^n_{yy} = -\beta^2 U^{n-2} \qquad U^n(0, y) = U^n(\xi, 0) = U^n(\xi, 1) = 0 \quad \text{(A25)}$$

$$+ \text{matching condition}$$

where n is an even number greater than or equal to two

To leading order, the boundary layer expansion is

$$U \sim 1 + \sum_{m=1}^{\infty} b_m \, e^{-m\pi\xi} \sin m\pi y + 0(\eta^2) \quad \text{(A26)}$$

where the b_m are the Fourier cofficients of $g(y) - 1$.

Since the boundary layer construction is the same near $x = 1$, we can write the composite expansion as [20]:

$$u \sim 1 + \sum_{m}^{\infty} b_m \, e^{-m\pi\xi_1} \sin m\pi y + \sum_{m=1}^{\infty} b_m \, e^{-m\pi\xi_2} \sin m\pi y + 0(\eta^2) \quad \text{(A27)}$$

where

$$\xi_1 = x/\eta \quad \text{(A28)}$$

and

$$\xi_2 = (1 - x)/\eta \quad \text{(A29)}$$

It is clear from eqn (A27) that whatever the form of $g(y)$, its effect decays exponentially as we move away from the ends of the rectangle.

Since $u = w/\Delta$ and $\eta = d/2$ we have that $w \sim \Delta$ in the rectangle and deviates from that only in the boundary layer of thickness $0(d/2)$ near the ends of the rectangle. We interpret this in terms of the title problem as being a good approximation to rigid body motion of the material contained between the two rigid line inclusions that represent the deep, narrow, rectangular foundation.